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NUMERICAL MODELING OF FILLING OF AN AXISYMMETRIC CHANNEL WITH A NONLINEAR VISCOPLASTIC FLUID WITH ALLOWANCE FOR THE Π-EFFECT

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The finite-element method is applied to describe the slow flow ($Re << 1$) of a nonlinear-viscoplastic fluid described by the Shul'man rheological model, which is implemented on filling of vertical volumes in the gravity field. The sliding effect of the fluid near a solid wall (the Π-effect) is taken into consideration. Numerical studies of the process of filling of an axisymmetric region are carried out. An influence of the Π-effect on the hydrodynamic process of filling is shown.

Many of the highly filled polymer compositions under melt processing manifest near-wall and orientation effects (the Π-effect) [1-4]. As the strain intensity grows, the polymer composition begins to slide near a solid wall with a velocity proportional to shear stresses developed in this region:

$$\tau_{\text{fr}} = -\varphi_{\text{sl.fr}} U_{\text{sl}}^{s} ,$$

τ_{fr} is the friction stress, $\varphi_{\text{sl.fr}}$ is the coefficient of sliding friction; U_{sl} is the sliding velocity on solid walls; s is the nonlinearity parameter of the Π-effect.

The present study is devoted to mathematical modeling of a nonlinear viscoplastic fluid flow with a free surface implemented on filling of axisymmetric volumes with low shear rates in the presence of the Π-effect.

In [5-8] the results of calculations of a non-Newtonian fluid flow with a free surface with allowance for abnormal motion near solid surfaces are reported. However, at present comprehensive investigations of the influence of the Π-effect on fluid flow dynamics of a nonlinear viscoplastic fluid with a free surface are unknown.

1. We consider a slow ($\text{Re} << 1$) flow of a nonlinear viscoplastic fluid with a free surface which is implemented on filling of the region between two coaxial cylinders (Fig. 1). A mathematical formulation of this problem in a quasistationary statement for a cylindrical coordinate system ($x_1 = r$; $x_3 = z$) will include:

the equations of motion

$$-\nabla_i P + \nabla_j \tau_{ij} = f_i , \quad i, j = 1, 3 ; \tag{1}$$

the discontinuity equation

$$\nabla_i U_i = 0 . \tag{2}$$

As a rheological equation, we adopt the Shul'man model [9]:

$$\tau_{ij} = 2\mu e_{ij} \quad \text{at} \quad \tau_{\text{II}} > \tau_0 ; \quad e_{ij} = 0 \quad \text{at} \quad \tau_{\text{II}} \le \tau_0 , \tag{3}$$

where a non-Newtonian fluid is described by

$$\mu = [\tau_0^{1/n} + (\mu_{\text{p}} A)^{1/m}]^n A^{-1} . \tag{4}$$

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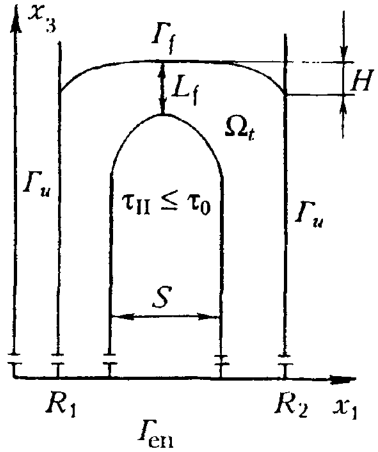


Fig. 1. Geometry and boundaries of the region to be filled.

The problem (1) through (4) will be solved under the following boundary conditions:

a) at the entrance to the flow region Γ_{en} the profile of the steady-state fluid flow with the prescribed rheology (3), (4) is given as

$$U_1 = 0 , \quad U_3 = U_3 (\mu, x_k) ; \tag{5}$$

b) on the free surface of the fluid Γ_f the dynamic and kinematic boundary conditions are specified as follows

$$\tau_{ij} n_j - (P - P_{ext}) n_i = 0 , \quad \frac{dx_i}{dt} = U_i , \quad i = 1, 3 . \tag{6}$$

On solid walls Γ_u of the time-variable calculated region Ω_t, the following boundary conditions are adopted that allow for the П-effect [7, 8]:

$$U_1 = 0 , \quad \begin{cases} U_3 = 0 , & \text{if} \quad |\tau_{13}| \le \tau_p , \\ |\tau_{13}| - \tau_p = \eta_{fr} U_3 , & \text{if} \quad |\tau_{13}| > \tau_p . \end{cases} \tag{7}$$

Here τ_p and η_{fr} are the empirical constants of sliding friction.

This boundary condition needs explanation. With increasing strain intensity, in a polymeric viscoplastic medium the "skeleton" formed by the filler undergoes complete rupture, while in the vicinity of the solid wall a narrow layer of binder is formed, whose viscosity is lower by several orders of magnitude than that of the main structure of the material. This indicates that the medium near the solid wall begins to slide. According to (7), on the solid wall Γ_u the adhesion condition is satisfied until the shear stresses that develop on the wall exceed the limit of τ_p, after which the fluid starts sliding over the wall with a velocity proportional to the shear stress.

As the initial conditions of the problem, the fields of the velocity, pressure and non-Newtonian fluid obtained in solving the problem (1)-(7) with a plane free surface were adopted and the nonstationary problem on filling of a region by a Shul'man nonlinear viscoplastic fluid reduces to a sequence of solutions of quasistationary problems on each time layer.

2. Numerically the problem (1)-(7) is solved by the finite-element method in a weak Galerkin formulation that makes it possible to satisfy, in a natural way, the dynamic boundary conditions on a free surface (6) and the sliding friction condition (7) in the form of an integral over the boundary. The initial problem is solved in a combined formulation in terms of the velocity and pressure variables [7, 8]. To obtain a stable solution and to determine exactly the velocity and pressure fields, a different order of approximation of the quantities sought is used. Thus, the second order of approximation is used for velocity, while for pressure the first order is adopted.

A finite-element grid is generated in the time-variable calculated region in an automatic mode by an algebraic method with the use of adaptive algorithms [8] which allow the finite-element grid to thicken in the region of strong gradients of a solution and singularities. For the problem under consideration such regimes are: 1) the

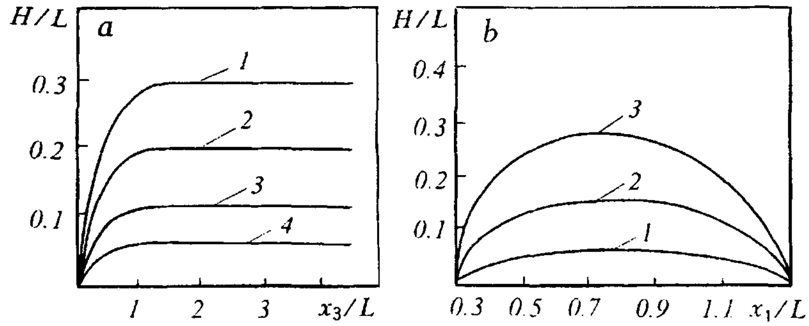


Fig. 2. Evolution of the maximum bending of the free surface as a function of Tr_1 (a): 1) $\mathrm{Tr}_1 = 10^4$; 2) 10^2; 3) 10; 4) 0 at $\mathrm{Tr}_2 = 12$ and a profile of the free surface vs. the parameter Tr_2 (b): 1) $\mathrm{Tr}_2 = 5$; 2) 12; 3) > 26 at $\mathrm{Tr}_1 =$ 10.

"boundary" of the fluid-quasisolid body transition, where an abrupt change in the non-Newtonian viscosity occurs, and 2) the region in the vicinity of the line of contact of the free fluid surface with the solid walls (the line of three-phase contact (LTPC)).

To eliminate undifferentiable singularities with the conditions of sliding friction on walls (7) being satisfied and with performing straight-through count in the quasisolid flow regions ($\tau_{II} \le \tau_0$) as the strain-rate intensity $A \to 0$ and the viscosity $\mu \to \infty$, the Shul'man rheological model (3), (4) and boundary condition (7) were modified by introducing the small parameter [10]:

$$\mu_\varepsilon = [\tau_0^{1/n} + (\mu_p A_\varepsilon)^{1/m}]^n A_\varepsilon^{-1}, \quad A_\varepsilon = \sqrt{2e_{ij}e_{ji} + \varepsilon^2}; \tag{8}$$

$$U_1 = 0, \quad |\tau_{13}| = \varphi_{\mathrm{sl.fr}} U_3, \quad \varphi_{\mathrm{sl.fr}} = \tau_p/\sqrt{U_3^2 + \xi^2} + \eta_{\mathrm{fr}}, \tag{9}$$

where $\varepsilon \ll 1$ and $\xi \ll 1$ are regularization parameters. In the calculations, they were assumed to be $\varepsilon = \xi = 10^{-8}$.

Application of the weak formulation of the finite-element method to Eqs. (1)-(2) reduces the initial differential problem to a system of the nonlinear projective-grid equations

$$\int_{\Omega_t} \left[- K_\gamma P_\gamma \nabla_i N_\alpha + \mu_\varepsilon (\nabla_i N_\beta U_{j\beta} + \nabla_j N_\beta U_{i\beta}) \nabla_j N_\alpha\right] d\Omega +$$

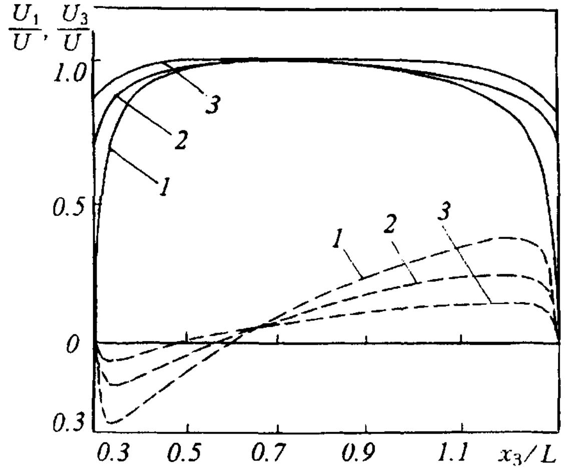
$$+ \int_{\Omega_t} f_i N_\alpha \, d\Omega + \int_{\Gamma_f} P_{\mathrm{ext}} n_i N_\alpha \, d\Omega - \int_{\Gamma_u} \varphi_{\mathrm{sl.fr}} U_3 N_\alpha t_i \, d\Omega = 0, \tag{10}$$

$$\int_{\Omega_t} \nabla_i N_\beta U_{i\beta} K_\gamma = 0, \quad i, j = 1, 3; \quad \alpha, \beta = \overline{1, 9}; \quad \gamma = \overline{1, 4}. \tag{11}$$

Here η_i, t_i are the normal and the tangent to the surface, respectively; N_α, K_γ are the basis functions for velocity and pressure approximations.

The system of nonlinear projective-grid equations (10), (11) is solved by the Newton–Rafson method [11]. For visualization of motion of the free surface, particles-markers are placed on it. Their position at the next time step is determined by integration of the kinematic boundary condition (6). The marker positions found are used to approximate the free surface by cubic splines. Then in the newly obtained region the finite-element grid is completely constructed.

3. Let us consider some results of the calculations. The axisymmetric region Ω is filled at a constant flow rate. For convenience of analysis, we introduce the dimensionless complexes: $\mathrm{St} = \mathrm{Re}/\mathrm{Fr}$ is the Stokes parameter, $\mathrm{Bin} = \tau_0/(\mu_p/A_m)$ is the Bingham parameter, and the parameters characterizing the ratio of the viscous forces and the sliding friction effect: $\mathrm{Tr}_1 = \eta_{\mathrm{fr}} L/\mu_{\mathrm{eff}}$ and $\mathrm{Tr}_2 = \tau_p/(\mu_p A_m)$.



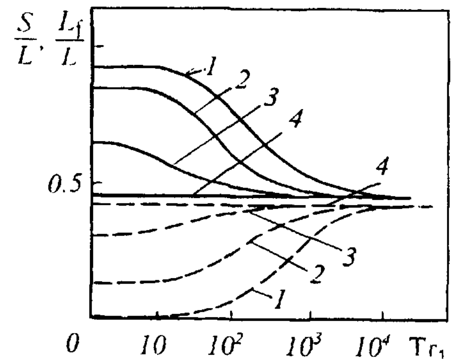


Fig. 3. Profiles of the axial (solid lines) and radial (dashed) velocities on the free surface at $Tr_2 = 12$: 1) $Tr_1 = 10^4$; 2) 100; 3) 0.

Fig. 4. Widths of the quasisolid flow core S (solid lines) and dimensions of the spouting flow region L_f (dashed) as a function of Tr_1 and Tr_2: 1) $Tr_2 = 0$; 2) 12; 3) 20; 4) > 26.

Calculations were made at the following values of the parameters: St = 20, Bin = 10, $\mu_p = 10^3$ Pa·sec, $n = 0.95$, $m = 1$, $U = 10^{-3}$ m/sec, $A_m = 10^{-3}$ sec^{-1}, $R_1/R_2 = 0.3/1.3$, $Tr_1 = 0-10^5$, $Tr_2 = 0-30$.

The results calculated have shown that from some moment of time the free fluid surface acquires a steady form and moves in an axial direction at the mean flow rate of the main flow. Figure 2 illustrates the influence of the parameters Tr_1 and Tr_2 on free surface bending and the process of establishing the free surface profile. An increase in Tr_1 from infinity to zero entails an increase in the sliding velocity of the nonlinear viscoplastic fluid over the solid wall. This leads to a decrease in the free surface bending and a reduction in the ways it is established (Fig. 2a). An increase in Tr_2 at a fixed Tr_1 causes an increase in bending of the form of the free fluid surface established (Fig. 2b).

Figure 3 shows the influence of the П-effect on the profiles of axial and radial velocities on the free fluid surface at different Tr_1 values. As is seen, as Tr_1 increases, the axial velocity profile on the free surface becomes more filled, while the radial component values decrease.

A characteristic feature of this hydrodynamic process is the presence of two zones in the fluid flow: 1) a region of a spouting two-dimensional stream in the vicinity of the free surface and 2) a region of the one-dimensional main stream at a distance from the moving front of the free surface. One more feature of the flow of viscoplastic fluids is the presence of a quasisolid flow zone in the main stream, where the intensity of stresses is lower than the yield strength of the fluid ($\tau_0 \geq \tau_{II}$). For this hydrodynamic process it represents a hollow cylinder with a "sharpening" in the vicinity of the moving front of the free surface (Fig. 1). Investigations of the influence of the main rheodynamic parameters of the process of filling on the size and location of the spouting flow region and of the quasisolid zone are carried out in [8, 12, 13]. Figure 4 illustrates the influence of the П-effect on the size and location of the quasisolid stream zone in the spouting flow region. An analysis of the calculated results reveals that an increase in the Tr_1 parameters leads to narrowing of the quasisolid flow zone and to an increase in the dimensions of the spouting flow region. This is a consequence of the intensity increasing of the shear stresses in the main flow and in the vicinity of the moving free surface of the viscoplastic fluid. A change in the Tr_2 values exerts a particularly strong influence on the hydrodynamic process at small Tr_1. Its increase causes growth of the spouting flow region and an increase in the width of the quasisolid flow core. At large Tr_1, i.e., $Tr_1 > 10^4$ the parameter Tr_2 does not influence the process of filling.

As is seen from Fig. 4, the curves show pronounced asymptotics. An analysis of the calculated results has shown that at $Tr_1 > 10^4$ the maximum axial component of the velocity vector on the solid walls of the region is less than 10^{-6} of the mean flow rate, which can be seen as fulfillment of the adhesion condition. Here, the parameter Tr_2 does not exert any influence on the process of filling.

If $Tr_1 = 0$, then for all values of $Tr_2 \le$ Bin filling is carried out in the presence of a plane free surface, i.e., the sliding velocity is equal to the mean flow rate and the quasisolid flow core occupies the entire width of the flow channel.

NOTATION

U_1, U_3, radial and axial components of the vector velocity; P, pressure; Ω, axisymmetric region to be filled; Ω_t,calculated region at the moment of time t; t, time; Γ_f, free fluid surface; Γ_u, solid walls of the region; Γ_{en}, entrance to the flow region; $f_i = (0, -\rho g)$, vector of mass forces; ρ, fluid density; g, free fall acceleration; τ_{ij}, tension of viscous stresses; $e_{ij} = (\nabla_i U_j + \nabla_j U_i)/2$, tensor of strain rates; τ_0, yield strength of the fluid; μ_p, plastic viscosity; n, m, constants of the Shul'man rheological model; $\tau_{II} = \sqrt{\tau_{ij}\tau_{ji}/2}$, $A = \sqrt{2e_{ij}e_{ji}}$, intensity of strain stresses and rates, respectively; P_{ext}, prescribed external pressure (above the free surface); n_i, t_i, vector components of the normal and the tangent to the surface; $Re = \rho UL/\mu_{eff}$, $Fr = U^2/(gL)$, Reynolds and Froude numbers; $L = R_2 - R_1$, characteristic size; R_1, R_2, radii of the internal and external cylinders; $\mu_{eff} = [\tau_0^{1/n} + (\mu_p A_m)^{1/m}]^n A_m^{-1}$, effective viscosity; $A_m = U/L$, mean intensity of strain rates; U, mean flow rate; H, maximum bending of the free surface; S, width of the quasisolid flow core; L_f, dimension of the spouting flow region; τ_p and η_{fr}, empirical coefficients of the friction model; N_α, K_γ, basic functions of the velocity and pressure approximations; Tr_1, Tr_2, dimensionless parameters of the Π-effect.

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